10th Real Number Solved problems

1. Prove that if a and b are integers, with $b > 0$, then there exist unique integers q and r satisfying:

$$a = qb + r \quad \text{with } 2b \le r < 3b$$

Observe: The Division Algorithm guarantees that if a and b are integers, with $b > 0$, then there exist unique integers q' and r' satisfying:

$$a = q'b + r' \quad \text{with } 0 \le r' < b$$

If we define $r = r' + 2b$, then $2b \le r < 3b$.

The trick now, is to define q such that $a = qb + r \quad$ with $2b \le r < 3b$.

To do this, we start with the relationship guaranteed by the Division Algorithm, namely:

$$a = q'b + r' \quad \text{with } 0 \le r' < b$$

Since $r = r' + 2b$ (or equivalently, $r' = r - 2b$), we can substitute $r - 2b$ for r'. This yields:

$$a = q'b + (r - 2b) \quad \text{with } 2b \le r < 3b$$
$$a = (q' - 2)\,b + r \quad \text{with } 2b \le r < 3b$$

This suggests that we let $q = q' - 2$. This yields:

$$a = qb + r \quad \text{with } 2b \le r < 3b$$

2. Show that any integer of the form $6k + 5$ is also of the form $3j + 2$, but not conversely.

Let $n = 6k + 5$. Then $n = 6k + 5 = 3\,(2k) + 5 = 3\,(2k) + 3 + 2 = 3\,(2k + 1) + 2$.

Thus, $n = 6k + 5 = 3j + 2$, where $j = 2k + 1$.

To show that the converse does NOT hold, let $n = 3j + 2$.

For $j = 2$, we have $n = 3\,(2) + 2 = 8$

If $n = 3j + 2 = 6k + 5$, then $n = 3j + 2 = 8 = 6k + 5$.

But $6k + 5 = 8 \Rightarrow 6k = 3 \Rightarrow k = \frac{1}{2}$, which is not an integer.

Hence, for $j = 2$, $n = 3j + 2 \neq 6k + 5$

3. Use the Division Algorithm to establish the following:

(a) The square of any integer is either of the form $3k$ or $3k + 1$.

Let n be an integer. By the Division Algorithm, either

$$n = 3m$$
$$n = 3m + 1$$
$$n = 3m + 2$$

If $n = 3m$, then $n^2 = (3m)^2 = 9m^2 = 3(3m^2) = 3k$, for $k = 3m^2$

If $n = 3m+1$, then $n^2 = (3m+1)^2 = 9m^2+6m+1 = 3(3m^2+2m)+1 = 3k+1$, for $k = 3m^2+2m$

If $n = 3m+2$, then $n^2 = (3m+2)^2 = 9m^2+12m+4 = 9m^2+12m+3+1 = 3(3m^2+4m+1)+1 = 3k+1$, for $k = 3m^2+4m+1$.

Hence, for any integer n, n^2 is either of the form $3k$ or $3k+1$.

(b) The cube of any integer has one of the forms, $9k$, $9k+1$, or $9k+8$.
Let n be an integer. By the Division Algorithm, either

$$n = 3m$$
$$n = 3m+1$$
$$n = 3m+2$$

If $n = 3m$, then $n^3 = (3m)^3 = 27m^3 = 9(3m^3) = 9k$, for $k = 3m^3$

If $n = 3m+1$, then $n^3 = (3m+1)^3 = 27m^3+27m^2+9m+1 = 9(3m^3+3m^2+m)+1 = 9k+1$, for $k = 3m^3+3m^2+m$

If $n = 3m+2$, then $n^3 = (3m+2)^3 = 27m^3+54m^2+36m+8 = 9(2m^3+2m^2+4m)+8 = 9k+8$, for $k = 2m^3+2m^2+4m$

Hence, for any integer n, n^3 has one of the forms, $9k$, $9k+1$, or $9k+8$.

(c) The fourth power of any integer is either of the form $5k$ or $5k+1$.
Let n be an integer. By the Division Algorithm, either

$$n = 5m$$
$$n = 5m+1$$
$$n = 5m+2$$
$$n = 5m+3$$
$$n = 5m+4$$

If $n = 5m$, then $n^4 = (5m)^4 = 625m^4 = 5(125m^4) = 5k$, for $k = 125m^4$

If $n = 5m+1$, then $n^4 = (5m+1)^4 = 625m^4+500m^3+150m^2+20m+1 = 5(125m^4+100m^3+30m^2+4m)+1 = 5k+1$, for $k = 125m^4+100m^3+30m^2+4m$

If $n = 5m+2$, then $n^4 = (5m+2)^4 = 625m^4+1000m^3+600m^2+160m+16 =$
$625m^4+1000m^3+600m^2+160m+15+1 =$
$5(125m^4+200m^3+125m^2+32m+3)+1 =$
$5k+1$, for $k = 125m^4+200m^3+125m^2+32m+3$

If $n = 5m+3$, then $n^4 = (5m+3)^4 = 625m^4+1500m^3+1350m^2+540m+81 =$
$625m^4+1500m^3+1350m^2+540m+80+1 =$
$5(125m^4+300m^3+270m^2+108m+16)+1 =$
$5k+1$, for $k = 125m^4+300m^3+270m^2+108m+16$

If $n = 5m+4$, then $n^4 = (5m+4)^4 = 625m^4 + 2000m^3 + 2400m^2 + 1280m + 256 =$
$625m^4 + 2000m^3 + 2400m^2 + 1280m + 255 + 1 =$
$5(125m^4 + 400m^3 + 480m^2 + 256m + 51) + 1 =$
$5k+1$, for $k = 125m^4 + 400m^3 + 480m^2 + 256m + 51$

Hence, for any integer n, n^4 is either of the form $5k$ or $5k+1$.

4. Prove that $3a^2 - 1$ is never a perfect square.

Observe that $3a^2 - 1 = 3(a^2 - 1) + 2 = 3k + 2$, for $k = a^2 - 1$.

The results of problem 3.a tell us that the square of an integer must either be of the form $3k$ or $3k+1$. Hence, $3a^2 - 1 = 3k + 2$ cannot be a perfect square.

5. For $n \geq 1$, prove that $n(n+1)(2n+1)/6$ is an integer.

Let n be an integer. By the Division Algorithm, either

$$\begin{aligned} n &= 6m \\ n &= 6m+1 \\ n &= 6m+2 \\ n &= 6m+3 \\ n &= 6m+4 \\ n &= 6m+5 \end{aligned}$$

If $n = 6m$, then $n(n+1)(2n+1)/6 = 6m(6m+1)(2(6m)+1)/6 =$
$(432m^3 + 108m^2 + 6m)/6 = 72m^3 + 18m^2 + m$

i.e., $n(n+1)(2n+1)/6$ is an integer, for $n = 6m$

If $n = 6m+1$, then $n(n+1)(2n+1)/6 = (6m+1)[(6m+1)+1][2(6m+1)+1]/6 =$
$(432m^3 + 324m^2 + 78m + 6)/6 = 72m^3 + 54m^2 + 13m + 1$

i.e., $n(n+1)(2n+1)/6$ is an integer, for $n = 6m+1$

If $n = 6m+2$, then $n(n+1)(2n+1)/6 = (6m+2)[(6m+2)+1][2(6m+2)+1]/6 =$
$(432m^3 + 540m^2 + 222m + 30)/6 = 72m^3 + 90m^2 + 37m + 5$

i.e., $n(n+1)(2n+1)/6$ is an integer, for $n = 6m+2$

If $n = 6m+3$, then $n(n+1)(2n+1)/6 = (6m+3)[(6m+3)+1][2(6m+3)+1]/6 =$
$(432m^3 + 756m^2 + 438m + 84)/6 = 72m^3 + 126m^2 + 73m + 14$

i.e., $n(n+1)(2n+1)/6$ is an integer, for $n = 6m+3$

If $n = 6m+4$, then $n(n+1)(2n+1)/6 = (6m+4)[(6m+4)+1][2(6m+4)+1]/6 =$
$(432m^3 + 972m^2 + 726m + 180)/6 = 72m^3 + 162m^2 + 121m + 30$

i.e., $n(n+1)(2n+1)/6$ is an integer, for $n = 6m+4$

If $n = 6m+5$, then $n(n+1)(2n+1)/6 = (6m+5)[(6m+5)+1][2(6m+5)+1]/6 =$
$(432m^3 + 1188m^2 + 1086m + 330)/6 = 72m^3 + 198m^2 + 181m + 55$

i.e., $n(n+1)(2n+1)/6$ is an integer, for $n = 6m+5$

Thus, $n(n+1)(2n+1)/6$ is an integer for all integers, n.

6. Show that the cube of any integer is of the form $7k$ or $7k \pm 1$.

Let n be an integer. By the Division Algorithm, either

$n = 7k$ $\quad n = 7k + 1$ $\quad n = 7k + 2$ $\quad n = 7k + 3$

$n = 7k + 4$ $\quad n = 7k + 5$ $\quad n = 7k + 6$

If $n = 7m$, then $n^3 = (7m)^3 = 343m^3 = 7(49m^3) = 7k$.

Hence, if $n = 7m$, then $n^3 = 7k$, for $k = 49m^3$

If $n = 7m + 1$, then $n^3 = (7m + 1)^3 = 343m^3 + 147m^2 + 21m + 1 =$
$7(49m^3 + 21m^2 + 3m) + 1 = 7k + 1$.

Hence, if $n = 7m + 1$, then $n^3 = 7k + 1$, for $k = 49m^3 + 21m^2 + 3m$

If $n = 7m + 2$, then $n^3 = (7m + 2)^3 = 343m^3 + 294m^2 + 84m + 8 =$
$343m^3 + 294m^2 + 84m + 7 + 1 = 7(49m^3 + 42m^2 + 12m + 1) + 1 = 7k + 1$.

Hence, if $n = 7m + 1$, then $n^3 = 7k + 1$, for $k = 49m^3 + 42m^2 + 12m + 1$

If $n = 7m + 3$, then $n^3 = (7m + 3)^3 = 343m^3 + 441m^2 + 189m + 27 =$
$343m^3 + 441m^2 + 189m + 28 - 1 = 7(49m^3 + 63m^2 + 27m + 4) - 1 = 7k - 1$.

Hence, if $n = 7m + 3$, then $n^3 = 7k - 1$, for $k = 49m^3 + 63m^2 + 27m + 4$

If $n = 7m + 4$, then $n^3 = (7m + 4)^3 = 343m^3 + 588m^2 + 336m + 64 =$
$343m^3 + 588m^2 + 336m + 63 + 1 = 7(49m^3 + 84m^2 + 48m + 9) + 1 = 7k + 1$.

Hence, if $n = 7m + 4$, then $n^3 = 7k + 1$, for $k = 49m^3 + 84m^2 + 48m + 9$

If $n = 7m + 5$, then $n^3 = (7m + 5)^3 = 343m^3 + 735m^2 + 525m + 125 =$
$343m^3 + 735m^2 + 525m + 126 - 1 = 7(49m^3 + 105m^2 + 75m + 18) - 1 = 7k - 1$.

Hence, if $n = 7m + 5$, then $n^3 = 7k - 1$, for $k = 49m^3 + 105m^2 + 75m + 18$

If $n = 7m + 6$, then $n^3 = (7m + 6)^3 = 343m^3 + 882m^2 + 756m + 216 =$
$343m^3 + 882m^2 + 756m + 217 - 1 = 7(49m^3 + 126m^2 + 108m + 31) - 1 = 7k - 1$.

Hence, if $n = 7m + 5$, then $n^3 = 7k - 1$, for $k = 49m^3 + 126m^2 + 108m + 31$

Hence, the cube of any integer is of the form $7k$ or $7k \pm 1$.

7. Prove that no integer in the following sequence is a perfect square:

$$11, 111, 1111, 11111, \ldots$$

First, observe that the first term, 11, is not a perfect square.

Next. observe that after the first term of the sequence, a typical term, $111\ldots111$, can be written as

$$111\ldots108 + 3 = 4k + 3$$

By an earlier observation, any perfect square fits either the form $4k$ or the form $4k+1$. Hence, no term in the sequence can be a perfect square.

8. Prove that 4^{th} Power of an odd integer is expressible in the form of 16n + 1 for natural number n

Solution If m is odd, we can write $m = 2k + 1$ for some $k \in \mathbb{Z}$ and so

$$m^4 = (2k+1)^4 = 16k^4 + 32k^3 + 24k^2 + 8k + 1$$

and so we have

$$m = 16\left(k^4 + 2k^3 + \left(\frac{k(3k+1)}{2}\right)\right) + 1 \quad = \quad 16n + 1$$

9. Prove that every positive integer different from 1 can be expressed as a product of nonnegative power of 2 and an odd number

Solution: Any odd number can be written as the product of a non-negative power of 2 and an odd number.

Let n be an odd number. Then $n = 2^0 \times n = 1 \times n = n$

For even numbers, we can follow the Fundamental Theorem of Arithmetic. The theorem states that every integer can be written as the product of prime numbers.

It says that an even number n will have 2 as a prime factor.

Let n be an even number where n is a power of two and can be written as $n = 2^k \times 1$, since 1 is an odd number, and the prime factorization of $n = (2^k \times p_1 \times p_2 \times \ldots\ldots\ldots p_n)$, where k is some positive integer, and $p_1, p_2 \ldots\ldots\ldots, p_n$ are primes. Since $p_1, p_2 \ldots\ldots\ldots, p_n$ are prime numbers greater than two, they are odd.

Therefore their product will also be odd. Thus n can be written as the product of a power of two and an odd number

10. Let a, b, c and d be positive rationals such that a + √b = c + √d , then either a = c and b = d or b and d are squares of rational numbers.

Solution: Given:

Case: (i) if a = c => a + √b = c + √d will be written as √b = √d => b = d

Case(ii) if $a \neq c$ then let a = c + k where k is rational number and $k \neq 0$

- ⇨ => a + √b = c + √d will be written as c + k + √b = c + √d
- ⇨ k + √b = √d
- ⇨ Squaring both sides we get $(k + \sqrt{b})^2 = (\sqrt{d})^2$
- ⇨ $K^2 + b + 2k\sqrt{b} = d$
- ⇨ $\sqrt{b} = \frac{d - k^2 - b}{2k}$
- ⇨ Rational number = Irrational number

Hence, √b is irrational .This is only possible only when b is square of rational number

Thus, d is also the square of rational number as k + √b = √d

11. Write the HCF and LCM of the smallest odd composite number and the smallest odd prime number.

Solution:

Smallest odd prime number = 3

Smallest odd composite number = 9

H.C.F. of (3, 9) = 3

L.C.M. of (3, 9) = 9

12. If an odd number p divides q^2 then will it divide q^3 also. State the reason

Solution: Given an odd number p divides q^2.

⇒ p is a factor of q^2 or $\frac{q^2}{p}$ gives 0 as a remainder.

Now, $q^2 \times q = q^3$, and p is a factor of q^2,

Hence it will also be a factor of $q^2 \times q$.

⇒ p is a factor of q^3

Therefore, p will divide q^3 also.

In general, we can say that if p divides q^2 then it will also divide q^n, where n is a natural number ($n \geq 2$).

13. Show that n^2 leaves the remainder 1 when divided by 8, where n is an odd positive integer.

Solution: Let n = 4a + 1

$n^2 = (4a + 1)^2 = 16a^2 + 8a + 1 = 8(a^2 + a) + 1 = 8k + 1$

14. Prove that if n is odd then $n^2 – 1$ is divisible by 8.

Solution: Let n = 4a + 1

$n^2 - 1 = (4a + 1)^2 - 1 = 16a^2 + 8a + 1 - 1 = 8(a^2 + a) = 8k$ where $k = a^2 + a$

15. Pens are sold in pack of 8 and notepads are sold in pack of 12. Find the least number of pack of each type that one should buy so that there are equal number of pen and notepads [CBSE 2014]

Solution: LCM of 8 and 12 is 24;

The least number of pack the pack of pen = $\frac{22}{8}$ = 3 ;

The least number of pack the pack of note book = pen = $\frac{24}{12} = 2$

16. A boy with collection of marbles realizes that if he makes a group of 5 or 6 marbles at time there are always two marbles left. Can you explain why the boy can’t have prime numbers of merles?

Solution: Given that, He makes a group of 5 or 6 marbles at time there are always two marbles left.

According to Euclid’s Division lemma: a = bq + r

Then, Numbers of marble = $5m + 2 \; or \; 6n + 2$.

As remainder is 2, when it is divided by $5 \; or \; 6$.

Thus, number of marbles must be = $(the \; multiple \; of \; 5 \times 6) + 2$

=> $30m + 2 = 2(15m + 1)$ = even number.

So, it cannot be a prime number

17. A school library has 280 science journals and 300 math’s journals. Students were told to stack these journals in such a way that each stack contains equal number of journals. Determine the number of stacks of science and maths journals. What is the benefit of library in student life?

Solution: Since, each stack contains equal number of journals.

=> Number of journals in each stack = HCF (280, 300) = 20

=> Number of stacks of science journals = 280/20 = 14

Number of stacks of maths journals = 300/20 = 15 ,

Library helps students to develop life-long learning skills.