Chapter 7
Hamilton's Principle - Lagrangian and Hamiltonian Dynamics

Many interesting physics systems describe systems of particles on which many forces are acting. Some of these forces are immediately obvious to the person studying the system since they are externally applied. Other forces are not immediately obvious, and are applied by the external constraints imposed on the system. These forces are often difficult to quantify, but the effect of these forces is easy to describe. Trying to describe such a system in terms of Newton's equations of motion is often difficult since it requires us to specify the total force. In this Chapter we will see that describing such a system by applying Hamilton's principle will allow us to determine the equation of motion for system for which we would not be able to derive these equations easily on the basis of Newton's laws. We should stress however, that Hamilton's principle does not provide us with a new physical theory, but it allows us to describe the existing theories in a new and elegant framework.

Hamilton's Principle

The evolution of many physical systems involves the minimization of certain physical quantities. We already have encountered an example of such a system, namely the case of refraction where light will propagate in such a way that the total time of flight is minimized. This same principle can be used to explain the law of reflection: the angle of incidence is equal to the angle of reflection.

The minimization approach to physics was formalized in detail by Hamilton, and resulted in Hamilton's Principle which states:

> " *Of all the possible paths along which a dynamical system may more from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.* "

We can express this principle in terms of the calculus of variations:

$$\delta \int_{t_1}^{t_2} (T - U)\, dt = 0$$

The quantity T - U is called the Lagrangian L.

Consider first a single particle, moving in a conservative force field. For such a particle, the kinetic energy T will just be a function of the velocity of the particle, and the potential energy will just be a function of the position of the particle. The Lagrangian is thus also a function of the position and the velocity of the particle. Hamilton's theorem states that we need to minimize the Lagrangian and thus require that

$$\delta \int_{t_1}^{t_2} L(x_i, \dot{x}_i)\, dt = 0$$

In Chapter 6 we have developed the theory required to solve problems of this type and found that the Lagrangian must satisfy the following relation:

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0$$

These last equations are called the **Lagrange equations of motion.** Note that in order to generate these equations of motion, we do not need to know the forces. Information about the forces is included in the details of the kinetic and potential energy of the system.

Consider the example of a plane pendulum. For this system, there is only one coordinate we need to specify, namely the polar angle θ. The kinetic energy T of the pendulum is equal to

$$T = \frac{1}{2} ml^2 \dot{\theta}^2$$

and the potential energy U is given by

$$U = mgl(1 - \cos\theta)$$

The Lagrangian for this system is thus equal to

$$L = T - U = \frac{1}{2} ml^2 \dot{\theta}^2 - mgl(1 - \cos\theta)$$

The equation of motion can now be determined and is found to be equal to

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -mgl\sin\theta - \frac{d}{dt}\left(ml^2\dot{\theta}\right) = -mgl\sin\theta - ml^2\ddot{\theta} = 0$$

or

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

This equation is of course the same equation we can find by applying Newton's force laws. I this example, the only coordinate that was used was the polar angle θ. Even though the pendulum is

a 3-dimensional system, the constraints imposed upon its motion reduced the number of degrees of freedom from 3 to 1.

Generalized Coordinates

If we try to describe a system of n particles, we need in general $3n$ coordinates to specify the position of its components. If external constraints are imposed on the system, the number of degrees of freedom may be less. If there are m constraints applied, the number of degrees of freedom will be $3n - m$. The coordinates do not need to be the coordinates of a coordinate system, but can be any set of quantities that completely specifies the state of the system. The state of the system is thus full specified by a point in the configuration space (which is a $3n - m$ dimensional space). The time evolution of the system can be described by a path in the configuration space.

The generalized coordinates of a system are written as $q_1, q_2, q_3, \ldots\ldots$ The generalized coordinates are of course related to the physical coordinates of the particles being described:

$$x_{\alpha,i} = x_{\alpha,i}\left(q_1, q_2, q_3, ..., t\right) = x_{\alpha,i}\left(q_j, t\right)$$

where $i = 1, 2, 3$ and $\alpha = 1, 2, ..., n$. Since the generalized coordinates in general will depend on time, we can also introduce the generalized velocities. The physical velocities will depends on the generalized velocities:

$$\dot{x}_{\alpha,i} = \dot{x}_{\alpha,i}\left(q_j, \dot{q}_j, t\right)$$

Equations of Motion in Generalized Coordinates

Based on the introduction of the Lagrangian and generalized coordinates, we can rephrase Hamilton's principle in the following way:

> " *Of all the possible paths along which a dynamical system may more from one point to another in configuration space within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the Lagrangian function for the system.* "

Thus

$$\delta \int_{t_1}^{t_2} L\left(q_i, \dot{q}_i, t\right) dt = 0$$

and

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

When we use the Lagrange's equations to describe the evolution of a system, we must recognize that these equations are only correct of the following conditions are met:

1. the force acting on the system, except the forces of constraint, must be derivable from one or more potentials.
2. the equations of constraint must be relations that connect the coordinates of the particles, and may be time dependent (note: this means that they are independent of velocity).

Constraints that do not depend on velocity are called **holonomic constraints**. There are two different types of holonomic constraints:

1. fixed or **scleronomic constraints**: constraints that do not depend on time.
2. moving or **rheonomic constraints**: constraints that depend on time.

Example: Problem 7.4

A particle moves in a plane under the influence of a force $f = -Ar^{\alpha-1}$ directed toward the origin; A and α are constants. Choose appropriate generalized coordinates, and let the potential energy be zero at the origin. Find the Lagrangian equations of motion. Is the angular momentum about the origin conserved? Is the total energy conserved?

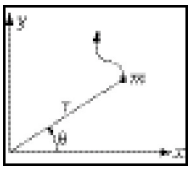

Figure 1. Problem 7.4.

If we choose (r,θ) as the generalized coordinates, the kinetic energy of the particle is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{7.4.1}$$

Since the force is related to the potential by

$$f = -\frac{\partial U}{\partial r} \tag{7.4.2}$$

we find

$$U = \frac{A}{\alpha} r^{\alpha} \tag{7.4.3}$$

where we let $U(r = 0) = 0$. Therefore, the Lagrangian becomes

$$L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{A}{\alpha} r^{\alpha} \tag{7.4.4}$$

Lagrange's equation for the coordinate r leads to

$$m\ddot{r} - mr\dot{\theta}^2 + Ar^{\alpha-1} = 0 \tag{7.4.5}$$

Lagrange's equation for the coordinate θ leads to

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \tag{7.4.6}$$

Since $mr^2\dot{\theta} = \ell$ is identified as the angular momentum, (7.4.6) implies that angular momentum is conserved. Now, if we use ℓ, we can write (7.4.5) as

$$m\ddot{r} - \frac{\ell^2}{mr^3} + Ar^{\alpha-1} = 0 \tag{7.4.7}$$

Multiplying (7.4.7) by $\dot{r}$, we have

$$m\dot{r}\ddot{r} - \frac{\dot{r}\ell^2}{mr^3} + A\dot{r}\, r^{\alpha-1} = 0 \tag{7.4.8}$$

which is equivalent to

$$\frac{d}{dt}\left[\frac{1}{2} m\dot{r}^2\right] + \frac{d}{dt}\left[\frac{\ell^2}{2mr^2}\right] + \frac{d}{dt}\left[\frac{A}{\alpha} r^{\alpha}\right] = 0 \tag{7.4.9}$$

Therefore,

$$\frac{d}{dt}(T+U)=0 \tag{7.4.10}$$

and the total energy is conserved.

Example: Problem 7.8

Consider a region of space divided by a plane. The potential energy of a particle in region 1 is U_1 and in region 2 it is U_2. If a particle of mass m and with speed v_1 in region 1 passes from region 1 to region 2 such that its path in region 1 makes and angle θ_1 with the normal to the plane of separation and an angle θ_2 with the normal when in region 2, show that

$$\frac{\sin\theta_1}{\sin\theta_2}=\sqrt{1+\frac{U_1-U_2}{T_1}}$$

where $T_1 = (1/2)mv_1^2$,

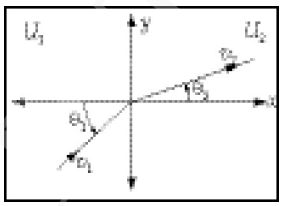

Figure 2. Problem 7.8.

Let us choose the (x, y) coordinates so that the two regions are divided by the y axis:

$$U(x)=\begin{cases} U_1 & x<0 \\ U_2 & x>0 \end{cases}$$

If we consider the potential energy as a function of x as above, the Lagrangian of the particle is

$$L=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)-U(x) \tag{7.8.1}$$

Therefore, Lagrange's equations for the coordinates x and y are

$$m\ddot{x} + \frac{dU(x)}{dx} = 0 \tag{7.8.2}$$

$$m\ddot{y} = 0 \tag{7.8.3}$$

Using the relation

$$m\ddot{x} = \frac{d}{dt}\, m\dot{x} = \frac{dp_x}{dt} = \frac{dp_x}{dx}\,\frac{dx}{dt} = \frac{p_x}{m}\,\frac{dp_x}{dx} \tag{7.8.4}$$

(7.8.2) becomes

$$\frac{p_x}{m}\,\frac{dp_x}{dx} + \frac{dU(x)}{dx} = 0 \tag{7.8.5}$$

Integrating (7.8.5) from any point in the region 1 to any point in the region 2, we find

$$\int_1^2 \frac{p_x}{m}\,\frac{dp_x}{dx}\,dx + \int_1^2 \frac{dU(x)}{dx}\,dx = 0 \tag{7.8.6}$$

$$\frac{p_{x_2}^2}{2m} - \frac{p_{x_1}^2}{2m} + U_2 - U_1 = 0 \tag{7.8.7}$$

or, equivalently,

$$\frac{1}{2}\, m\dot{x}_1^2 + U_1 = \frac{1}{2}\, m\dot{x}_2^2 + U_2 \tag{7.8.8}$$

Now, from (7.8.3) we have

$$\frac{d}{dt}\, m\dot{y} = 0$$

and $m\dot{y}$ is constant. Therefore,

$$m\dot{y}_1 = m\dot{y}_2 \tag{7.8.9}$$

From (7.8.9) we have

$$\frac{1}{2} m\dot{y}_1^2 = \frac{1}{2} m\dot{y}_2^2 \tag{7.8.10}$$

Adding (7.8.8) and (7.8.10), we have

$$\frac{1}{2} mv_1^2 + U_1 = \frac{1}{2} mv_2^2 + U_2 \tag{7.8.11}$$

From (7.8.9) we also have

$$mv_1 \sin\theta_1 = mv_2 \sin\theta_2 \tag{7.8.12}$$

Substituting (7.8.11) into (7.8.12), we find

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1} = \left[1 + \frac{U_1 - U_2}{T_1}\right]^{1/2} \tag{7.8.13}$$

This problem is the mechanical analog of the refraction of light upon passing from a medium of a certain optical density into a medium with a different optical density.

Lagrange's Equations with Undetermined Multipliers

We have seen already a number of examples were one could remove the equations of constraint by a suitable choice of coordinates. For example, when we looked at the motion of an object on the surface of a cylinder we could either:

1. Use a set of three coordinates to describe the motion, coupled with one equation of constraint.
2. Use a set of two coordinates (such at the azimuthal angle and the vertical position) to describe the motion, without an equation of constraint.

In this Section we will look at situations where the constraint depends on the velocity:

$$f\left(x_{\alpha,i}, \dot{x}_{\alpha,i}, t\right) = 0$$

If the constraints can be expressed in a differential form,

$$\sum_{j=1}^{s} \frac{\partial f_k}{\partial q_j} dq_j = 0$$

we can directly incorporate it into the Lagrange equations:

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} + \sum_{k=1}^{m} \lambda_k(t) \frac{\partial f_k}{\partial q_j} = 0$$

It turns out that the forces of constraint can be determined from the equations of constraints and the Lagrange multipliers $\lambda_m(t)$:

$$Q_j = \sum_{k=1}^{m} \lambda_k(t) \frac{\partial f_k}{\partial q_j}$$

where Q_j is the j^{th} component of the generalized force, expressed in generalized coordinates.

The use of Lagrange multiplier to determine the forces of constraints is nicely illustrated in Example 7.9 in the textbook, where a disk rolling down an inclined plane is being studied. If the disk does not slip, we find that the distance along the plane y and the angle of rotation θ are related, and the equation of constraint is

$$f(y,\theta) = y - R\theta = 0$$

The textbook explains in detail how the Lagrange equations are solved in this case, and I will not reproduce this here. The solution shows us that the Lagrange multiplier is given by

$$\lambda = -\frac{Mg\sin\alpha}{3}$$

By combining the equation of constraint and the Lagrange multipliers we can determine the generalized forces of constraint:

$$Q_y = \lambda(t)\frac{\partial f}{\partial y} = -\frac{Mg\sin\alpha}{3}$$

and

$$Q_\theta = \lambda(t)\frac{\partial f}{\partial \theta} = \frac{MgR\sin\alpha}{3}$$

Note that these forces of constraint do not have to be all pure forces. The force of constraint associated with y is the friction force between the disk and the plane that is required to ensure that the disk rolls without slipping. However, the force of constraint associated with the angle θ is the torque of this friction force with respect to the center of the disk. We need to note the generalized force does not have to have the unit of force.

It is also important to note that if we had chosen to solve the problem by expressing the Lagrangian in terms of a single coordinate y, by eliminating the angle, we would not have obtained any information about the forces of constraint. Although I have stressed that in many cases, you can simplify the solution by the proper choice of coordinates such that the equations of constraint are eliminated, in this case, the solution will not provide any information about the forces of constraint.

Example: Problem 7.12

A particle of mass m rests on a smooth plane. The plane is raised to an inclination angle θ at a constant rate α ($\theta = 0°$ at $t = 0$), causing the particle to move down the plane. Determine the motion of the particle.

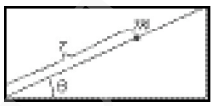

Figure 3. Problem 7.12.

This problem is an example of a problem with a velocity-dependent constraint. However, if we can easily incorporate the constraint into the Lagrangian, we do not need to worry about constraint functions. In this example, we use our knowledge of the constraint immediately in our expression of the kinetic and the potential energy. Putting the origin of our coordinate system at the bottom of the plane we find

$$L = T - U = \frac{1}{2} m \left(\dot{r}^2 + r^2 \, \dot{\theta}^2 \right) - mgr \sin\theta$$

$$\theta = \alpha t; \quad \dot{\theta} = \alpha$$

$$L = \frac{1}{2} m \left(\dot{r}^2 + \alpha^2 r^2 \right) - mgr \sin \alpha t$$

Lagrange's equation for r gives

$$m\ddot{r} = m\alpha^2 r - mg \sin \alpha t$$

or

$$\ddot{r} - \alpha^2 r = -g \sin \alpha t \tag{7.12.1}$$

The general solution is of the form $r = r_h + r_n$ where r_h is the general solution of the homogeneous equation $\ddot{r} - \alpha^2 r = 0$ and r_n is a particular solution of Eq. (7.12.1). So

$$r_h = Ae^{\alpha t} + Be^{-\alpha t}$$

For r_n, try a solution of the form $r_p = C \sin \alpha t$. Then $\ddot{r}_p = -C\,\alpha^2 \sin \alpha t$. Substituting into (7.12.1) gives

$$-C\,\alpha^2 \sin \alpha t - C\ \alpha^2 \sin \alpha t = -g \sin \alpha t$$

$$C = \frac{g}{2\alpha^2}$$

So

$$r(t) = Ae^{\alpha t} + Be^{-\alpha t} + \frac{g}{2\alpha^2} \sin \alpha t$$

We can determine A and B from the initial conditions:

$$r(0) = r_0 \tag{7.12.2}$$

$$\dot{r}(0) = 0 \tag{7.12.3}$$

Equation (7.12.2) implies:

$$r_0 = A + B$$

Equation (7.12.3) implies:

$$0 = A - B + \frac{g}{2\alpha^2}$$

Solving for A and B gives:

$$A = \frac{1}{2}\left[r_0 - \frac{g}{2\alpha^2} \right] \qquad B = \frac{1}{2}\left[r_0 + \frac{g}{2\alpha^2} \right]$$

$$r(t) = \frac{1}{2}\left[r_0 - \frac{g}{2\alpha^2} \right] e^{\alpha t} + \frac{1}{2}\left[r_0 + \frac{g}{2\alpha^2} \right] e^{-\alpha t} + \frac{g}{2\alpha^2} \sin \alpha t$$

or

$$r(t) = r_0 \cosh \alpha t + \frac{g}{2\alpha^2} (\sin \alpha t - \sinh \alpha t)$$

Although we have found an analytical solution to this problem, we need to examine if the solution matches our expectation of the motion of the mass m. The best way to do this is to plot a graph of the motion of the mass in a Cartesian coordinate system. Consider the situation where r_0 = 10 m. Figure 4 shows the trajectory of the mass for two different value of α: α = 0.1 rad/s and α = 0.03 rad/s.

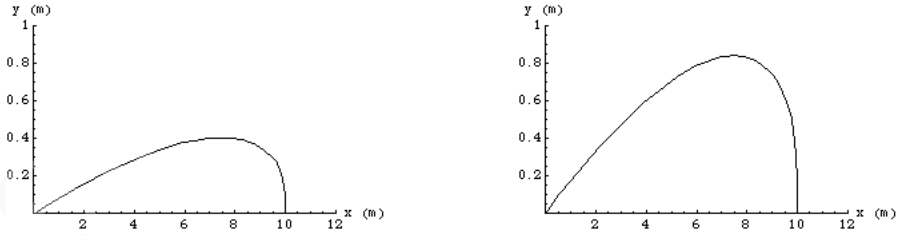

Figure 4. Solution of Problem 7.12 with r_0 = 10 m and α = 0.01 rad/s (left) and α = 0.03 rad/s (right).

The generalized momentum

One of the big differences between the equations of motion obtained from the Lagrange equations and those obtained from Newton's equations is that in the latter case, the coordinate

frame used is always a Cartesian coordinate frame. When we use the Lagrange equations we have the option to choice generalized coordinates that do not have to correspond to the coordinates of a Cartesian coordinate system.

The generalized coordinates are related to the Cartesian coordinates, and transformation rules allow use to carry out transformations between coordinate systems. The generalized forces of constraint are related to the Newtonian forces of constraint, as was illustrated in Example 7.9 in the textbook. The similarities between the Cartesian and the generalized parameters suggest it may also be useful to consider the concept of the generalized momentum.

In a Cartesian coordinate system we can easily determine the connection between the Lagrangian and the linear momentum. The Lagrangian is equal to

$$L = T - U = \frac{1}{2} m \sum_{i=1}^{3} \dot{x}_i^{\,2} - U(x_i)$$

The Lagrange equation for this Lagrangian is given by

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = -\frac{\partial U}{\partial x_i} - \frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} = 0$$

and we can rewrite this as

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} = -\frac{\partial U}{\partial x_i} = F_i = m\ddot{x}_i = \frac{d}{dt} p_i$$

This last equation suggest that we define the generalized momentum of a particle in the following way:

$$p_i = \frac{\partial T}{\partial \dot{x}_i}$$

It is obviously consistent with our definition of linear momentum in Cartesian coordinates.

Consider a particle moving in a two-dimensional plane and having its motion described in terms of spherical coordinates. The kinetic energy of the particle is equal to

$$T = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Since there are two generalized coordinates we can determine two generalized momenta:

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r}$$

which is the linear momentum of the particle, and

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

which is the angular momentum of the particle. We thus see that two distinct concepts from our introductory courses emerge directly from our Lagrangian theory.

Homogeneous functions

Consider a homogeneous quadratic function f that depends only on the products of the generalized velocities:

$$f = \sum_{j,k} a_{j,k}\dot{q}_j\dot{q}_k$$

An example of such function would be the kinetic energy of a particle. Consider what happens when we differentiate this function with respect to one of the generalized velocities:

$$\frac{\partial f}{\partial \dot{q}_l} = \sum_j a_{j,l}\dot{q}_j + \sum_k a_{l,k}\dot{q}_k$$

If we multiply this equation by dq_l/dt and sum over all values of l we obtain:

$$\sum_l \dot{q}_l \frac{\partial f}{\partial \dot{q}_l} = \sum_l \dot{q}_l \left(\sum_j a_{j,l}\dot{q}_j + \sum_k a_{l,k}\dot{q}_k \right) = \sum_{l,j} a_{j,l}\dot{q}_j\dot{q}_l + \sum_{l,k} a_{l,k}\dot{q}_k\dot{q}_l = 2\sum_{j,k} a_{j,k}\dot{q}_j\dot{q}_k = 2f$$

In general, if f is a homogeneous function of the parameter y_k^n, then

$$\sum_k q_k \frac{\partial f}{\partial q_k} = nf$$

Conservation of Energy

If we consider a closed system, a system that does not interact with its environment, then we expect that the Lagrangian that describes this system does not depend explicitly on time. That is

$$\frac{\partial L}{\partial t} = 0$$

Of course, this does not mean that $dL/dt = 0$ since

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_k \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_j + \frac{\partial L}{\partial t} = \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_k \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_j$$

Using the Lagrange equations, we can rewrite this equation as

$$\frac{dL}{dt} = \sum_j \dot{q}_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_k \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_j = \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right)$$

This equation can be written as

$$\frac{d}{dt}\left(L - \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right)\right) = 0$$

or

$$L - \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = \text{constant} = -H$$

The constant H is called the Hamiltonian of the system and the Hamiltonian is defined as

$$H = \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) - L$$

The Hamiltonian H is a conserved quantity for the system we are currently considering. If we use Cartesian coordinates we find that

$$H = \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) - L = 2T - (T - U) = (T + U) = E$$

In this case we find that the Hamiltonian of the system is equal to the total energy of the system, and we thus conclude that the total energy is conserved. The equality of H and E is only satisfied if the following conditions are met:

- The potential U depends only on position, and not on velocity.
- The transformation rules connecting Cartesian and generalized coordinates are independent of time.

The latter condition is not met in for example a moving coordinate system, and in such system, the Hamiltonian will not be equal to the total energy.

We thus conclude that if the Lagrangian of a system does not depend explicitly on time, the total energy of that system will be conserved.

Example Problem 7.22

A particle of mass *m* moves in one dimension under the influence of a force *F*:

$$F(x,t) = \frac{k}{x^2} e^{-(t/\tau)}$$

where κ and τ are positive constants. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy and discuss the conservation of energy for the system.

The potential energy *U* corresponding to this force *F* must satisfy the relation

$$F = -\frac{\partial U}{\partial x}$$

and *U* must thus be equal to

$$U = \frac{k}{x} e^{-t/\tau}$$

Therefore, the Lagrangian is

$$L = T - U = \frac{1}{2} m\dot{x}^2 - \frac{k}{x} e^{-t/\tau}$$

The Hamiltonian is given by

$$H = p_x \dot{x} - L = \dot{x} \frac{\partial L}{\partial \dot{x}} - L$$

so that

$$H = \frac{p_x^2}{2m} + \frac{k}{x} e^{-t/\tau}$$

The Hamiltonian is equal to the total energy, $T + U$, because the potential does not depend on velocity, but the total energy of the system is not conserved because H contains the time explicitly.

Conservation of Linear Momentum

The Lagrangian should be unaffected by a translation of the entire system in space, assuming that space is homogeneous (which is one of the requirement of an inertial reference frame). Consider what happens when we carry out an infinitesimal displacement of the coordinate system along one of the coordinate axes. The change in the Lagrangian as a result of this displacement must be equal to zero:

$$\delta L = \frac{\partial L}{\partial x_i}\delta x_i + \frac{\partial L}{\partial \dot{x}_i}\delta \dot{x}_i = 0$$

We can rewrite this equation as

$$\delta L = \frac{\partial L}{\partial x_i}\delta x_i + \frac{\partial L}{\partial \dot{x}_i}\delta \dot{x}_i = \frac{\partial L}{\partial x_i}\delta x_i + \frac{\partial L}{\partial \dot{x}_i}\left(\frac{d}{dt}\delta x_i\right) = \frac{\partial L}{\partial x_i}\delta x_i = 0$$

Since the displacement is arbitrary, this equation can only be correct if

$$\frac{\partial L}{\partial x_i} = 0$$

Using the Lagrange equation this is equivalent to requiring

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0$$

or

$$\frac{\partial L}{\partial \dot{x}_i} = \text{constant}$$

Assuming that the potential U does not depend on velocity we see that this relation is equivalent to

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = p_i = \text{constant}$$

The consequence of the independence of the Lagrangian under a translation of space is that linear momentum is conserved.

Conservation of Angular Momentum

Space is an inertial reference frame is isotropic, which means that the properties of a system are unaffected by the orientation of the system. In this case we expect that the Lagrangian does not change when the coordinate axes are rotated through an infinitesimal angle. A rotation through such an angle produces the following change in the position vector:

$$\delta\bar{r} = \delta\bar{\theta} \times \bar{r}$$

The velocity vectors will change in the same way:

$$\delta\dot{\bar{r}} = \delta\bar{\theta} \times \dot{\bar{r}}$$

The Lagrangian should not change as a result of such a transformation. Thus we must require that

$$\delta L = \sum_i \left(\frac{\partial L}{\partial x_i} \delta x_i + \frac{\partial L}{\partial \dot{x}_i} \delta \dot{x}_i \right) = \sum_i \left\{ \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} \right) \delta x_i + \frac{\partial L}{\partial \dot{x}_i} \delta \dot{x}_i \right\} = \sum_i \left(\dot{p}_i \delta x_i + p_i \delta \dot{x}_i \right) = 0$$

We thus conclude that

$$\dot{\bar{p}} \bullet \delta\bar{r} + \bar{p} \bullet \delta\dot{\bar{r}} = 0$$

When we express the changes in terms of the rotation angle we obtain:

$$\dot{\bar{p}} \bullet \left(\delta\bar{\theta} \times \bar{r} \right) + \bar{p} \bullet \left(\delta\bar{\theta} \times \dot{\bar{r}} \right) = \delta\bar{\theta} \bullet \left[\dot{\bar{r}} \times \bar{p} + \bar{r} \times \dot{\bar{p}} \right] = \delta\bar{\theta} \bullet \left[\frac{d}{dt} \left(\bar{r} \times \bar{p} \right) \right] = 0$$

Since the angle of rotation was an arbitrary angle, this relation can only be satisfied if

$$\frac{d}{dt} \left(\bar{r} \times \bar{p} \right) = 0$$

or

$$\bar{r} \times \bar{p} = \bar{L} = \text{constant}$$

The angular momentum of the system is thus conserved. This conserved quantity is a direct consequence of the invariance of the Lagrangian for infinitesimal rotations. We conclude that the important conserved quantities are a direct consequence of the properties of the space (and its symmetries).

Canonical Equations of Motion

The Lagrangian we have discussed in this Chapter is a function of the generalized position and the generalized velocity. The equations of motion can also be expressed in terms of the generalized position and the generalized momentum. The generalized momentum is defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

We can use the generalized momentum to rewrite the Lagrange equations of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \dot{p}_i = \frac{\partial L}{\partial q_i}$$

The Hamiltonian can also be expressed in terms of the generalized momentum

$$H = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = \sum_j \dot{q}_j p_j - L$$

In general we will write the Hamiltonian in terms of the generalized position and the generalized momentum . The change in H due to small changes in time and in the generalized position and momentum is equal to

$$\begin{aligned} dH &= \sum_j \left(\dot{q}_j dp_j + p_j d\dot{q}_j - \frac{\partial L}{\partial q_j} dq_j - \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j \right) - \frac{\partial L}{\partial t} dt = \\ &= \sum_j \left(\dot{q}_j dp_j + p_j d\dot{q}_j - \frac{\partial L}{\partial q_j} dq_j - p_j d\dot{q}_j \right) - \frac{\partial L}{\partial t} dt = \\ &= \sum_j \left(\dot{q}_j dp_j - \frac{\partial L}{\partial q_j} dq_j \right) - \frac{\partial L}{\partial t} dt = \sum_j \left(\dot{q}_j dp_j - \dot{p}_j dq_j \right) - \frac{\partial L}{\partial t} dt \end{aligned}$$

The change in H can also be expressed in the following way:

$$dH = \sum_j \left(\frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial q_j} dq_j \right) + \frac{\partial H}{\partial t} dt$$

After combining the last two equations we obtained the following relation:

$$\sum_j \left(\frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial q_j} dq_j \right) + \frac{\partial H}{\partial t} dt = \sum_j \left(\dot{q}_j dp_j - \dot{p}_j dq_j \right) - \frac{\partial L}{\partial t} dt$$

or

$$\sum_j \left(\left\{ \frac{\partial H}{\partial p_j} - \dot{q}_j \right\} dp_j + \left\{ \frac{\partial H}{\partial q_j} + \dot{p}_j \right\} dq_j \right) + \left\{ \frac{\partial H}{\partial t} + \frac{\partial L}{\partial t} \right\} dt = 0$$

Since the variations in time and the generalized position and momenta are equal to independent, the coefficients of dq_i, dp_i, and dt must be zero. Thus:

$$\frac{\partial H}{\partial p_j} - \dot{q}_j = 0$$
$$\frac{\partial H}{\partial q_j} + \dot{p}_j = 0$$
$$\frac{\partial H}{\partial t} + \frac{\partial L}{\partial t} = 0$$

The first two equations are called **Hamilton's equations of motion** or the **canonical equations of motion**. Note:

- For each generalized coordinate there are two canonical equations of motion.
- For each generalized coordinate these is only one Lagrange equations of motion.
- Each canonical equation of motion is a first order differential equation.
- Each Lagrange equation of motion is a second order differential equation.

Although first order differential equations are in general easier to solve than second order differential equations, the Hamiltonian is often more difficult to construct than the Lagrangian since we must express the Hamiltonian in terms of the generalized position and the generalized momentum.

Example: Problem 7.38

The potential for an anharmonic oscillator is $U = kx^2/2 + bx^4/4$ where k and b are constants. Find Hamilton's equations of motion.

The Hamiltonian of the system is

$$H = T + U = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{kx^2}{2} + \frac{bx^4}{4} = \frac{p^2}{2m} + \frac{kx^2}{2} + \frac{bx^4}{4}$$

The Hamiltonian motion equations that follow this Hamiltonian are

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} = -(kx + bx^3)$$

Example: Problem 7.28.

The force F that is provided fixed the potential U:

$$U = -\frac{k}{r}$$

The Lagrangian, expressed in polar coordinates, is thus equal to

$$L = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{k}{r}$$

In order to use Hamilton's equations of motion we must express the Hamiltonian in terms of the generalized position and momentum. The following relations can be used to do this:

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \Rightarrow \quad \dot{r} = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{p_\theta}{mr^2}$$

Since the coordinate transformations are independent of t, and the potential energy is velocity-independent, the Hamiltonian is the total energy.

$$H = T + U = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{k}{r} =$$
$$= \frac{1}{2} m\left[\frac{p_r^2}{m^2} + r^2 \frac{p_\theta^2}{m^2 r^4}\right] - \frac{k}{r} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r}$$

Hamilton's equations of motion can now be found easily

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \qquad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$
$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{k}{r^2} \qquad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

Example: Problem 7.24.

Consider a simple plane pendulum consisting of a mass m attached to a string of length l. After the pendulum is set into motion, the length of the string is shortened at a constant rate:

$$\frac{dl}{dt} = -\alpha = \text{constant}$$

The suspension point remains fixed. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.

The kinetic energy and the potential energy of the system are expressed as

$$\left.\begin{aligned} T &= \frac{1}{2} m\left(\dot{\ell}^2 + \ell^2\dot{\theta}^2\right) = \frac{1}{2} m\left(\alpha^2 + \ell^2\dot{\theta}^2\right) \\ U &= -mg\ell\cos\theta \end{aligned}\right] \qquad (7.24.1)$$

The Lagrangian is equal to

$$L = T - U = \frac{1}{2} m\left(\alpha^2 + \ell^2\dot{\theta}^2\right) + mg\ell\cos\theta \qquad (7.24.2)$$

The Hamiltonian is

$$H = p_\theta \dot{\theta} - L = \frac{\partial L}{\partial \dot{\theta}} \dot{\theta} - L = \frac{p_\theta^2}{2m\ell^2} - \frac{1}{2} m\alpha^2 - mg\ell \cos\theta \tag{7.24.3}$$

which is different from the total energy, $T + U$. The total energy is thus not conserved in this system because work is done on the system and we have

$$\frac{d}{dt}(T+U) \neq 0 \tag{7.24.4}$$

NOTE: WE WILL SKIP SECTIONS 7.12 AND 7.13.